Exercise 4F

1 a $I_n = \int x^n e^{\frac{x}{2}} dx$

Let $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

Let $\frac{dv}{dx} = e^{\frac{x}{2}} \Rightarrow v = 2e^{\frac{x}{2}}$

$$\int x^n e^{\frac{x}{2}} dx = 2x^n e^{\frac{x}{2}} - \int 2nx^{n-1} e^{\frac{x}{2}} dx$$

$$= 2x^n e^{\frac{x}{2}} - 2n\int x^{n-1} e^{\frac{x}{2}} dx$$

Therefore:

$I_n = 2x^n e^{\frac{x}{2}} - 2nI_{n-1}$ as required

b Let $P_n = 2x^n e^{\frac{x}{2}}$

Therefore:

$$I_n = P_n - 2nI_{n-1}$$

$$I_3 = P_3 - 2\times 3I_2$$

$$= P_3 - 6I_2$$

$$= P_3 - 6[P_2 - 2\times 2I_1]$$

$$= P_3 - 6P_2 + 24I_1$$

$$= P_3 - 6P_2 + 24P_1 - 48I_0$$

$$\int x^3 e^{\frac{x}{2}} dx = 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 48\int e^{\frac{x}{2}} dx$$

$$= 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + c$$

2 a $I_n = \int_1^e x(\ln x)^n\, dx$

Let $u = (\ln x)^n \Rightarrow \frac{du}{dx} = \frac{n}{x}(\ln x)^{n-1}$

Let $\frac{dv}{dx} = x \Rightarrow v = \frac{1}{2}x^2$

$$\int_1^e x(\ln x)^n\, dx = \left[\frac{1}{2}x^2(\ln x)^n - \frac{1}{2}\int x^2 \times \frac{n}{x}(\ln x)^{n-1}\, dx\right]_1^e$$

$$= \left[\frac{1}{2}x^2(\ln x)^n\right]_1^e - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\, dx$$

$$= \frac{1}{2}e^2 - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\, dx$$

Therefore:

$I_n = \frac{1}{2}e^2 - \frac{n}{2}I_{n-1}$ as required

2 b $I_4 = \int_1^e x(\ln x)^4 \, dx$

$= \frac{1}{2}e^2 - \frac{4}{2}I_3$

$= \frac{1}{2}e^2 - 2\left(\frac{1}{2}e^2 - \frac{3}{2}I_2\right)$

$= \frac{1}{2}e^2 - e^2 + 3I_2$

$= -\frac{1}{2}e^2 + 3\left(\frac{1}{2}e^2 - \frac{2}{2}I_1\right)$

$= e^2 - 3I_1$

$= e^2 - 3\left(\frac{1}{2}e^2 - \frac{1}{2}I_0\right)$

$= -\frac{1}{2}e^2 + \frac{3}{2}I_0$

$= -\frac{1}{2}e^2 + \frac{3}{2}\int_1^e x(\ln x)^0 \, dx$

$= -\frac{1}{2}e^2 + \frac{3}{2}\int_1^e x \, dx$

$= -\frac{1}{2}e^2 + \frac{3}{2}\left[\frac{1}{2}x^2\right]_1^e$

$= -\frac{1}{2}e^2 + \frac{3}{4}(e^2 - 1)$

$= \frac{e^2}{4} - \frac{3}{4}$

$= \frac{e^2 - 3}{4}$ as required

3 $\int_0^1 (x+1)(x+2)\sqrt{1-x}\,dx$

$(x+1)(x+2) = x^2 + 3x + 2$

Therefore:

$$(x+1)(x+2)\sqrt{1-x} = (x^2+3x+2)\sqrt{1-x}$$
$$= x^2\sqrt{1-x} + 3x\sqrt{1-x} + 2\sqrt{1-x}$$

So:

$$\int_0^1 (x+1)(x+2)\sqrt{1-x}\,dx = \int_0^1 x^2\sqrt{1-x}\,dx + 3\int_0^1 x\sqrt{1-x}\,dx + 2\int_0^1 \sqrt{1-x}\,dx$$

From Example 21:

$$I_n = \int_0^1 x^n\sqrt{1-x}\,dx \Rightarrow I_n = \frac{2n}{2n+3}I_{n-1}$$

Hence:

$$\int_0^1 (x+1)(x+2)\sqrt{1-x}\,dx = I_2 + 3I_1 + 2I_0$$
$$= \frac{2\times 2}{(2\times 2+3)}I_1 + 3\times\frac{2}{2+3}I_0 + 2I_0$$
$$= \frac{4}{7}\times\frac{2}{2+3}I_0 + \frac{6}{5}I_0 + 2I_0$$
$$= \frac{8}{35}I_0 + \frac{6}{5}I_0 + 2I_0$$
$$= \frac{24}{7}I_0$$

$$I_0 = \int_0^1 x^0\sqrt{1-x}\,dx$$
$$= \int_0^1 \sqrt{1-x}\,dx$$

So:

$$\frac{24}{7}I_0 = \frac{24}{7}\int_0^1 \sqrt{1-x}\,dx$$
$$= \frac{24}{7}\left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1$$
$$= -\frac{16}{7}\left[(1-1)^{\frac{3}{2}} - (1-0)^{\frac{3}{2}}\right]$$
$$= \frac{16}{7}$$

4 a $I_n = \int x^n e^{-x} dx$

Let $I_n = u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

Let $\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\int x^n e^{-x} dx = -x^n e^{-x} + n\int x^{n-1} e^{-x} dx$

Therefore:

$I_n = -x^n e^{-x} + nI_{n-1}$ as required

b $I_3 = \int x^3 e^{-x} dx$

$= -x^3 e^{-x} + 3I_2$

$= -x^3 e^{-x} + 3\left(-x^2 e^{-x} + 2I_1\right)$

$= -x^3 e^{-x} - 3x^2 e^{-x} + 6I_1$

$= -x^3 e^{-x} - 3x^2 e^{-x} + 6\left(-x^1 e^{-x} + I_0\right)$

$= -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} + 6I_0$

$I_0 = \int x^0 e^{-x} dx$

$I_0 = \int e^{-x} dx$

Therefore:

$I_3 = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} + 6\int e^{-x} dx$

$I_3 = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} + 6e^{-x} + c$

4 c $I_n = -x^n e^{-x} + nI_{n-1}$

$I_4 = -x^n e^{-x} + 4I_3$

$= -x^n e^{-x} + 4\left(-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x}\right) + c$ using the result from part (b)

$= -x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24xe^{-x} - 24e^{-x} + c$

$= -e^{-x}\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right) + c$

$\int_0^1 x^4 e^{-x} dx = \left[-e^{-x}\left(x^4 + 4x^3 + 12x^2 + 24x + 24\right)\right]_0^1$

$= \left[-e^{-1}\left(1 + 4 + 12 + 24 + 24\right) + e^0\left(24\right)\right]$

$= 24 - \frac{65}{e}$

$= \frac{24e - 65}{e}$

5 a $I_n = \int \tanh^n x \, dx$

$= \int \tanh^{n-2} x \tanh^2 x \, dx$

$= \int \tanh^{n-2} x \left(1 - \operatorname{sech}^2 x\right) dx$

$= \int \tanh^{n-2} x \, dx - \int \tanh^{n-2} x \operatorname{sech}^2 x \, dx$

$= \int \tanh^{n-2} x \, dx - \frac{1}{n-1} \int \frac{d}{dx}\left(\tanh^{n-1} x\right) dx$

$= \int \tanh^{n-2} x \, dx - \frac{1}{n-1} \tanh^{n-1} x$

Therefore:

$I_n = I_{n-2} - \frac{1}{n-1} \tanh^{n-1} x$ as required

5 b $I_5 = \int \tanh^5 x \, dx$

$= I_3 - \frac{1}{5-1} \tanh^{5-1} x$

$= I_3 - \frac{1}{4} \tanh^4 x$

$= I_{3-2} - \frac{1}{3-1} \tanh^{3-1} x - \frac{1}{4} \tanh^4 x$

$= I_1 - \frac{1}{2} \tanh^2 x - \frac{1}{4} \tanh^4 x$

$I_1 = \int \tanh x \, dx$

Therefore:

$I_5 = \int \tanh x \, dx - \frac{1}{2} \tanh^2 x - \frac{1}{4} \tanh^4 x$

$= \ln\left(\cosh x\right) - \frac{1}{2} \tanh^2 x - \frac{1}{4} \tanh^4 x + c$

5 c $\int_0^{\ln 2} \tanh^4 x \, dx$

$I_4 = I_2 - \frac{1}{3}\tanh^3 x$

$= I_0 - \tanh x - \frac{1}{3}\tanh^3 x$

$I_0 = \int \tanh^0 x \, dx$

$= \int dx$

Therefore:

$I_4 = \int_0^{\ln 2} dx - \left[\tanh x + \frac{1}{3}\tanh^3 x\right]_0^{\ln 2}$

$= \left[x - \tanh x - \frac{1}{3}\tanh^3 x\right]_0^{\ln 2}$

Since:

$\tanh x = \frac{\sinh x}{\cosh x}$

$= \frac{\frac{1}{2}\left(e^x - e^{-x}\right)}{\frac{1}{2}\left(e^x + e^{-x}\right)}$

$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Therefore:

$\tanh 0 = \frac{e^0 - e^0}{e^0 + e^0}$ and $\tanh \ln 2 = \frac{e^{\ln 2} - e^{\ln 2}}{e^{\ln 2} + e^{\ln 2}} = \frac{2 - \frac{1}{2}}{2 + \frac{1}{2}} = \frac{3}{5}$

$I_4 = \left(\ln 2 - \frac{3}{5} - \frac{1}{3}\left(\frac{3}{5}\right)^3\right) - \left(0 - 0 - \frac{1}{3}(0)^3\right)$

$= \ln 2 - \frac{3 \times 5^2}{5^3} - \frac{3 \times 3}{5^3}$

$= \ln 2 - \frac{84}{125}$ as required

6 a $\int \tan^n x \, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \, dx$

Let $P_n = \frac{1}{n}\tan^n x$

$I_n = \int \tan^n x \, dx$

$= P_{n-1} - I_{n-2}$

Therefore:

$I_4 = P_3 - I_2$

$= P_3 - [P_1 - I_0]$

$= P_3 - P_1 + I_0$

$I_0 = \int \tan^0 x \, dx$

$= \int dx$

So

$I_4 = P_3 - P_1 + \int dx$

Hence:

$\int \tan^4 x \, dx = \frac{1}{3}\tan^3 x - \tan x + x + c$

b $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$

$I_5 = P_4 - I_3$

$= P_4 - [P_2 - I_1]$

$= P_4 - P_2 + I_1$

$I_1 = \int \tan x \, dx$

So

$I_5 = P_4 - P_2 + \int \tan x \, dx$

Hence:

$$\int_0^{\frac{\pi}{4}} \tan^5 x \, dx = \left[\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \left[\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln(\sec x)\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{4}\tan^4\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan^2\left(\frac{\pi}{4}\right) + \ln\left(\sec\left(\frac{\pi}{4}\right)\right)\right) - \left(\frac{1}{4}\tan^4 0 - \frac{1}{2}\tan^2 0 + \ln(\sec 0)\right)$$

$$= \frac{1}{4} - \frac{1}{2} + \ln\sqrt{2}$$

$$= \ln\sqrt{2} - \frac{1}{4}$$

6 c $\int_0^{\frac{\pi}{3}} \tan^6 x \, dx$

$$\begin{aligned} I_6 &= P_5 - I_4 \\ &= P_5 - [P_3 - I_2] \\ &= P_5 - P_3 + I_2 \\ &= P_5 - P_3 + [P_1 - I_0] \\ &= P_5 - P_3 + P_1 - I_0 \\ &= P_5 - P_3 + P_1 - \int dx \end{aligned}$$

Therefore:

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \tan^6 x \, dx &= \left[\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} dx \\ &= \left[\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x\right]_0^{\frac{\pi}{3}} \\ &= \left(\frac{1}{5}\tan^5\left(\frac{\pi}{3}\right) - \frac{1}{3}\tan^3\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right) - \left(\frac{1}{5}\tan^5 0 - \frac{1}{3}\tan^3 0 + \tan 0 - 0\right) \\ &= \frac{9\sqrt{3}}{5} - \frac{3\sqrt{3}}{3} + \sqrt{3} - \frac{\pi}{3} \\ &= \frac{9\sqrt{3}}{5} - \frac{\pi}{3} \text{ as required} \end{aligned}$$

7 a $I_n = \int_1^a (\ln x)^n \, dx$

Let $u = (\ln x)^n \Rightarrow \frac{du}{dx} = n(\ln x)^{n-1} \times \frac{1}{x}$

Let $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\begin{aligned} \int_1^a (\ln x)^n \, dx &= \left[x(\ln x)^n\right]_1^a - n\int_1^a x(\ln x)^{n-1} \times \frac{1}{x} dx \\ &= \left[x(\ln x)^n\right]_1^a - n\int_1^a (\ln x)^{n-1} dx \\ &= \left[a(\ln a)^n - 0\right] - n\int_1^a (\ln x)^{n-1} dx \\ &= a(\ln a)^n - n\int_1^a (\ln x)^{n-1} dx \end{aligned}$$

Therefore:

$I_n = a(\ln a)^n - nI_{n-1}$ as required

7 b $\int_1^2 (\ln x)^3 \, dx$

$$I_3 = a(\ln a)^3 - 3I_2$$
$$= a(\ln a)^3 - 3\left(a(\ln a)^2 - 2I_1\right)$$
$$= a(\ln a)^3 - 3a(\ln a)^2 + 6I_1$$
$$= a(\ln a)^3 - 3a(\ln a)^2 + 6\left(a(\ln a) - I_0\right)$$
$$= a(\ln a)^3 - 3a(\ln a)^2 + 6a\ln a - 6I_0$$

$$\int_1^a (\ln x)^0 \, dx = \int_1^a dx$$

$$\int_1^a (\ln x)^3 \, dx = a(\ln a)^3 - 3a(\ln a)^2 + 6a\ln a - 6\int_1^a dx$$
$$= a(\ln a)^3 - 3a(\ln a)^2 + 6a\ln a - 6[x]_1^a$$
$$= a(\ln a)^3 - 3a(\ln a)^2 + 6a\ln a - 6a + 6$$

Therefore:

$$\int_1^2 (\ln x)^3 \, dx = 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 12 + 6$$
$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6$$

7 c $\int_1^e (\ln x)^6 \, dx$

$I_n = a(\ln a)^n - nI_{n-1}$

Therefore:

$$\begin{aligned}
I_6 &= a(\ln a)^6 - 6I_5 \\
&= a(\ln a)^6 - 6\left(a(\ln a)^5 - 5I_4\right) \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30I_4 \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30\left(a(\ln a)^4 - 4I_3\right) \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120I_3 \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120\left(a(\ln a)^3 - 3I_2\right) \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360I_2 \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360\left(a(\ln a)^2 - 2I_1\right) \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360a(\ln a)^2 - 720I_1 \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360a(\ln a)^2 - 720\left(a(\ln a) - I_0\right) \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360a(\ln a)^2 - 720a(\ln a) + 720I_0 \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360a(\ln a)^2 - 720a(\ln a) + 720\int_1^a dx \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360a(\ln a)^2 - 720a(\ln a) + 720[x]_1^a \\
&= a(\ln a)^6 - 6a(\ln a)^5 + 30a(\ln a)^4 - 120a(\ln a)^3 + 360a(\ln a)^2 - 720a(\ln a) + 720a - 720
\end{aligned}$$

Therefore:

$$\begin{aligned}
\int_1^e (\ln x)^6 \, dx &= e(\ln e)^6 - 6e(\ln e)^5 + 30e(\ln e)^4 - 120e(\ln e)^3 + 360e(\ln e)^2 - 720e(\ln e) + 720e - 720 \\
&= e - 6e + 30e - 120e + 360e - 720e + 720e - 720 \\
&= 265e - 720 \\
&= 5(53e - 144)
\end{aligned}$$

8 a $\int_0^{\frac{\pi}{2}} \sin^7 x \, dx$

From Example 22:

$$I_n = \frac{(n-1)}{n} I_{n-2} \quad n \geq 2$$

$$I_7 = \frac{6}{7} I_5$$

$$= \frac{6}{7}\left(\frac{4}{5} I_3\right)$$

$$= \frac{24}{35} I_3$$

$$= \frac{24}{35}\left(\frac{2}{3} I_1\right)$$

$$= \frac{16}{35} I_1$$

$$= \frac{16}{35} \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \frac{16}{35} [-\cos x]_0^{\frac{\pi}{2}}$$

$$= \frac{16}{35}\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right)$$

$$= \frac{16}{35}$$

8 b $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx = \int_0^{\frac{\pi}{2}} \sin^2 x \left(1 - \sin^2 x\right)^2 dx$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \left(1 - 2\sin^2 x + \sin^4 x\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \, dx - 2\int_0^{\frac{\pi}{2}} \sin^4 x \, dx + \int_0^{\frac{\pi}{2}} \sin^6 x \, dx$$

$$= I_2 - 2I_4 + I_6$$

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$I_2 = \frac{1}{2} I_0$$

$$= \frac{\pi}{4}$$

$$I_4 = \frac{3}{4} I_2$$

$$= \frac{3\pi}{16}$$

$$I_6 = \frac{5}{6} I_4$$

$$= \frac{5\pi}{32}$$

Therefore:

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx = \frac{\pi}{4} - \frac{3\pi}{8} + \frac{5\pi}{32}$$

$$= \frac{\pi}{32}$$

8 c $\int_0^1 x^5\sqrt{1-x^2}\,dx$

Let $x=\sin\theta \Rightarrow \frac{dx}{d\theta}=\cos\theta$

When $x = 1$, $\theta=\frac{\pi}{2}$

When $x = 0$, $\theta=0$

$$\int_0^1 x^5\sqrt{1-x^2}\,dx=\int_0^1 \sin^5\theta\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$=\int_0^1 \sin^5\theta\cos^2\theta\,d\theta$$

$$=\int_0^1 \sin^5\theta\left(1-\sin^2\theta\right)d\theta$$

$$=\int_0^1 \sin^5\theta\,d\theta-\int_0^1 \sin^7\theta\,d\theta$$

$$=I_5-I_7$$

$$I_5=\frac{4}{5}I_3$$

$$I_5=\frac{4}{5}\left(\frac{2}{3}I_1\right)$$

$$=\frac{8}{15}I_1$$

$$I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx$$

$$=\left[-\cos x\right]_0^{\frac{\pi}{2}}$$

$$=-\cos x+\cos 0$$

$$=-\cos\left(\frac{\pi}{2}\right)+\cos 0$$

$$=1$$

Therefore:

$$I_5=\frac{8}{15}$$

$$I_7=\frac{6}{7}I_5$$

$$=\frac{16}{35}$$

So:

$$\int_0^1 x^5\sqrt{1-x^2}\,dx=\frac{8}{15}-\frac{16}{35}$$

$$=\frac{8}{105}$$

8 d $\int_0^{\frac{\pi}{6}} \sin^8 3t \, dt$

Let $u = 3t \Rightarrow \frac{du}{dt} = 3$

When $t = 0$, $u = 0$

When $t = \frac{\pi}{6}$, $u = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{6}} \sin^8 3t \, dt = \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^8 u \, du$$

$$= \frac{1}{3} I_8$$

$$I_8 = \frac{7}{8} I_6$$

$$= \frac{7}{8}\left(\frac{5}{6} I_4\right)$$

$$= \frac{35}{48} I_4$$

$$= \frac{35}{48}\left(\frac{3}{4} I_2\right)$$

$$= \frac{105}{192} I_2$$

$$\int_0^{\frac{\pi}{6}} \sin^8 3t \, dt = \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^8 u \, du$$

$$= \frac{1}{3} \times \frac{105}{192}\left(\frac{1}{2} I_0\right)$$

$$= \frac{35}{384} I_0$$

$$= \frac{35}{384}\left(\frac{\pi}{2}\right)$$

$$= \frac{35\pi}{768}$$

9 a $I_n = \int \frac{\sin^{2n} x}{\cos x} \mathrm{d}x$

$$I_{n+1} = \int \frac{\sin^{2(n+1)} x}{\cos x} \mathrm{d}x$$

$$= \int \frac{\sin^{2n+2} x}{\cos x} \mathrm{d}x$$

$$I_n - I_{n+1} = \int \frac{\sin^{2n} x}{\cos x} \mathrm{d}x - \int \frac{\sin^{2n+2} x}{\cos x} \mathrm{d}x$$

$$= \int \frac{\sin^{2n} x - \sin^{2n+2} x}{\cos x} \mathrm{d}x$$

$$= \int \frac{\sin^{2n} x - \left(\sin^{2n} x\right)\left(\sin^2 x\right)}{\cos x} \mathrm{d}x$$

$$= \int \frac{\sin^{2n} x\left(1 - \sin^2 x\right)}{\cos x} \mathrm{d}x$$

$$= \int \frac{\sin^{2n} x \cos^2 x}{\cos x} \mathrm{d}x$$

$$= \int \sin^{2n} x \cos x \, \mathrm{d}x$$

Using

$$\int f'(x)\left[f(x)\right]^n \mathrm{d}x = \frac{\left[f(x)\right]^n}{n+1} + c$$

gives:

$$\int \sin^{2n} x \cos x \, \mathrm{d}x = \frac{\sin^{2n+1} x}{2n+1}$$

Therefore:

$$I_n - I_{n+1} = \frac{\sin^{2n+1} x}{2n+1} \text{ as required}$$

9 b $I_{n+1} = \int \frac{\sin^4 x}{\cos x}\,dx$

$$I_n = \int \frac{\sin^2 x}{\cos x}\,dx$$
$$= \int \frac{1-\cos^2 x}{\cos x}\,dx$$
$$= \int \left(\frac{1}{\cos x} - \cos x\right)dx$$
$$= \int (\sec x - \cos x)\,dx$$

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

Let $u = \sec x + \tan x \Rightarrow \frac{du}{dx} = \sec x \tan x + \sec^2 x \Rightarrow du = \sec x(\tan x + \sec x)\,dx$

$$\int \sec x\,dx = \int \frac{1}{u}\,du$$
$$= \ln u$$
$$= \ln(\sec x + \tan x)$$

$$-\int \cos x\,dx = -\sin x$$

Therefore:

$$I_n = \ln(\sec x + \tan x) - \sin x$$

$$I_n - I_{n+1} = \frac{\sin^{2n+1} x}{2n+1}$$

When $n = 1$

$$I_n - I_{n+1} = \frac{\sin^3 x}{3}$$

$$I_{n+1} = I_n - \frac{\sin^3 x}{3}$$
$$= \ln(\sec x + \tan x) - \sin x - \frac{\sin^3 x}{3}$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos x}\,dx = \ln(\sec x + \tan x) - \sin x - \frac{\sin^3 x}{3}$$

$$= \ln\left(\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right) - \sin\left(\frac{\pi}{4}\right) - \frac{\sin^3\left(\frac{\pi}{4}\right)}{3}$$
$$= \ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}$$
$$= \ln\left(\sqrt{2}+1\right) - \frac{7\sqrt{2}}{12}$$

10 a $\int_0^1 x\left(1-x^3\right)^n \mathrm{d}x$

Let $u=(1-x^3)^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=-3nx^2(1-x^3)^{n-1}$

Let $\frac{\mathrm{d}v}{\mathrm{d}x}=x \Rightarrow v=\frac{1}{2}x^2$

Therefore:

$$\int_0^1 x\left(1-x^3\right)^n \mathrm{d}x=\left[\frac{1}{2}x^2\left(1-x^3\right)^n\right]_0^1+\frac{3n}{2}\int_0^1 x^4\left(1-x^3\right)^{n-1}\mathrm{d}x$$

$$\left[\frac{1}{2}x^2\left(1-x^3\right)^n\right]_0^1=0$$

Therefore:

$$\begin{aligned}\int_0^1 x\left(1-x^3\right)^n \mathrm{d}x&=\frac{3n}{2}\int_0^1 x^4\left(1-x^3\right)^{n-1}\mathrm{d}x\\&=\frac{3n}{2}\int_0^1 x\left(1-\left(1-x^3\right)\right)\left(1-x^3\right)^{n-1}\mathrm{d}x\\&=\frac{3n}{2}\int_0^1\left(x\left(1-x^3\right)^{n-1}-\left(1-x^3\right)\left(1-x^3\right)^{n-1}\right)\mathrm{d}x\\&=\frac{3n}{2}\int_0^1\left(x\left(1-x^3\right)^{n-1}-\left(1-x^3\right)^n\right)\mathrm{d}x\end{aligned}$$

$$\int_0^1 x\left(1-x^3\right)^n \mathrm{d}x=\frac{3n}{2}\int_0^1 x\left(1-x^3\right)^{n-1}\mathrm{d}x-\frac{3n}{2}\int_0^1 x\left(1-x^3\right)^n \mathrm{d}x$$

Hence:

$$I_n=\frac{3n}{2}I_{n-1}-\frac{3n}{2}I_n$$

$$I_n+\frac{3n}{2}I_n=\frac{3n}{2}I_{n-1}$$

$$I_n\left(1+\frac{3n}{2}\right)=\frac{3n}{2}I_{n-1}$$

$$I_n\left(\frac{2+3n}{2}\right)=\frac{3n}{2}I_{n-1}$$

$I_n=\frac{3n}{2+3n}I_{n-1}$ as required

10 b $I_4 = \frac{3\times4}{2+3\times4}I_3$

$= \frac{6}{7}I_3$

$= \frac{6}{7}\left(\frac{3\times3}{2+3\times3}I_2\right)$

$= \frac{54}{77}I_2$

$= \frac{54}{77}\left(\frac{3\times2}{2+3\times2}I_1\right)$

$= \frac{81}{154}I_1$

$= \frac{243}{770}I_0$

$I_0 = \int_0^1 x\left(1-x^3\right)^0 \mathrm{d}x$

$= \int_0^1 x \, \mathrm{d}x$

$= \left[\frac{1}{2}x^2\right]_0^1$

$= \frac{1}{2}$

Therefore:

$I_4 = \frac{243}{770}\times\frac{1}{2}$

$= \frac{243}{1540}$

11 a $I_n = \int_0^a \left(a^2 - x^2\right)^n \mathrm{d}x$

Let $u = \left(a^2 - x^2\right)^n \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2nx\left(a^2 - x^2\right)^{n-1}$

Let $\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$

$\int_0^a \left(a^2 - x^2\right)^n \mathrm{d}x = \left[x\left(a^2 - x^2\right)^n\right]_0^a + 2n\int_0^a x^2\left(a^2 - x^2\right)^{n-1} \mathrm{d}x$

$\left[x\left(a^2 - x^2\right)^n\right]_0^a = 0$

Therefore:

$$\begin{aligned}
\int_0^a \left(a^2 - x^2\right)^n \mathrm{d}x &= 2n\int_0^a x^2\left(a^2 - x^2\right)^{n-1} \mathrm{d}x \\
&= 2n\int_0^a \left(a^2 - \left(a^2 - x^2\right)\right)\left(a^2 - x^2\right)^{n-1} \mathrm{d}x \\
&= 2n\int_0^a \left(a^2\left(a^2 - x^2\right)^{n-1} - \left(a^2 - x^2\right)\left(a^2 - x^2\right)^{n-1}\right) \mathrm{d}x \\
&= 2n\int_0^a \left(a^2\left(a^2 - x^2\right)^{n-1} - \left(a^2 - x^2\right)^n\right) \mathrm{d}x \\
&= 2n\int_0^a a^2\left(a^2 - x^2\right)^{n-1} \mathrm{d}x - 2n\int_0^a \left(a^2 - x^2\right)^n \mathrm{d}x \\
&= 2na^2\int_0^a \left(a^2 - x^2\right)^{n-1} \mathrm{d}x - 2n\int_0^a \left(a^2 - x^2\right)^n \mathrm{d}x
\end{aligned}$$

Hence:

$I_n = 2na^2 I_{n-1} - 2nI_n$

$I_n + 2nI_n = 2na^2 I_{n-1}$

$I_n\left(1 + 2n\right) = 2na^2 I_{n-1}$

$I_n = \frac{2na^2}{1 + 2n} I_{n-1}$ as required

11 b i $I_4 = \frac{2\times 4a^2}{1+2\times 4} I_3$

$= \frac{8a^2}{9} I_3$

$= \frac{8a^2}{9}\left(\frac{2\times 3\times a^2}{1+2\times 3} I_2\right)$

$= \frac{16a^4}{21} I_2$

$= \frac{16a^4}{21}\left(\frac{2\times 2\times a^2}{1+2\times 2} I_1\right)$

$= \frac{64a^6}{105} I_1$

$= \frac{64a^6}{105}\left(\frac{2a^2}{1+2} I_0\right)$

$= \frac{128a^8}{315} I_0$

$I_0 = \int_0^a \left(a^2 - x^2\right)^0 \mathrm{d}x$

$= \int_0^a \mathrm{d}x$

$= [x]_0^a$

$= a$

Therefore:

$I_4 = \frac{128a^8}{315} \times a$

$= \frac{128a^9}{315}$

Since

$\int_0^a \left(a^2 - x^2\right)^4 \mathrm{d}x = \frac{128a^9}{315}$

$\int_0^1 \left(a^2 - x^2\right)^4 \mathrm{d}x = \frac{128}{315}$

11 b ii $\int_0^3 (9-x^2)^3 \, dx = \int_0^3 (3^2-x^2)^3 \, dx$

Using

$$I_n = \frac{2na^2}{1+2n} I_{n-1}$$

gives:

$$I_3 = \frac{2\times 3a^2}{1+2\times 3} I_2$$
$$= \frac{6a^2}{7} I_2$$
$$= \frac{6a^2}{7}\left(\frac{2\times 2a^2}{1+2\times 2} I_1\right)$$
$$= \frac{24a^4}{35} I_1$$
$$= \frac{24a^4}{35}\left(\frac{2a^2}{1+2} I_0\right)$$
$$= \frac{16a^6}{35} I_0$$

$$I_0 = \int_0^a (a^2-x^2)^0 \, dx$$
$$= \int_0^a dx$$
$$= [x]_0^a$$
$$= a$$

Therefore:

$$\int_0^a (a^2-x^2)^3 \, dx = \frac{16a^6}{35} \times a$$
$$= \frac{16a^7}{35}$$

and

$$\int_0^3 (3^2-x^2)^3 \, dx = \frac{16(3)^7}{35}$$
$$= \frac{34\,992}{35}$$

11 b iii $I_{\frac{1}{2}} = \int_0^2 \sqrt{4-x^2}\,dx$

$$I_{-\frac{1}{2}} = \int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \left[\arcsin\left(\frac{x}{2}\right)\right]_0^2$$

$$= \arcsin 1$$

Using

$$I_n = \frac{2na^2}{1+2n} I_{n-1}$$

gives:

$$I_{\frac{1}{2}} = \frac{2 \times \frac{1}{2} a^2}{1 + 2 \times \frac{1}{2}} I_{-\frac{1}{2}}$$

$$= \frac{a^2}{2} I_{-\frac{1}{2}}$$

$$= \frac{a^2}{2} \arcsin 1$$

Therefore:

$$\int_0^2 \sqrt{4-x^2}\,dx = \frac{2^2}{2}\arcsin 1$$

$$= 2\arcsin 1$$

$$= \pi$$

11 c Let $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta$

when $x = 2,\ \theta = \frac{\pi}{2}$

when $x = 0,\ \theta=0$

$$\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta} \times 2\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sqrt{4\left(1-\sin^2\theta\right)} \times \cos\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta} \times \cos\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(1+\cos 2\theta\right)d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right)$$

$$= \pi$$

12 a $I_n = \int_0^4 x^n \sqrt{4-x}\, dx$

Let $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

Let $\frac{dv}{dx} = \sqrt{4-x} \Rightarrow v = -\frac{2}{3}(4-x)^{\frac{3}{2}}$

$$\int_0^4 x^n \sqrt{4-x}\, dx = \left[-\frac{2}{3}x^n(4-x)^{\frac{3}{2}}\right]_0^4 + \frac{2}{3}n\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\, dx$$

$$\left[-\frac{2}{3}x^n(4-x)^{\frac{3}{2}}\right]_0^4 = 0$$

Therefore:

$$\begin{aligned}\int_0^4 x^n \sqrt{4-x}\, dx &= \frac{2}{3}n\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\, dx\\ &= \frac{2}{3}n\int_0^4 x^{n-1}(4-x)(4-x)^{\frac{1}{2}}\, dx\\ &= \frac{2}{3}n\int_0^4 \left[4x^{n-1}(4-x)^{\frac{1}{2}} - x\cdot x^{n-1}(4-x)^{\frac{1}{2}}\right] dx\\ &= \frac{2}{3}n\int_0^4 \left[4x^{n-1}(4-x)^{\frac{1}{2}} - x^n(4-x)^{\frac{1}{2}}\right] dx\\ &= \frac{8}{3}n\int_0^4 x^{n-1}\sqrt{4-x}\, dx - \frac{2}{3}n\int_0^4 x^n\sqrt{4-x}\, dx\end{aligned}$$

Hence:

$$I_n = \frac{8}{3}nI_{n-1} - \frac{2}{3}nI_n$$

$$I_n + \frac{2}{3}nI_n = \frac{8}{3}nI_{n-1}$$

$$3I_n + 2nI_n = 8nI_{n-1}$$

$$I_n = \frac{8n}{3+2n}I_{n-1} \text{ as required}$$

12 b $I_3 = \int_0^4 x^3\sqrt{4-x}\,dx$

$$I_3 = \frac{8\times 3}{3+2\times 3}I_2$$
$$= \frac{8}{3}I_2$$
$$= \frac{8}{3}\left(\frac{8\times 2}{3+2\times 2}I_1\right)$$
$$= \frac{128}{21}I_1$$
$$= \frac{128}{21}\left(\frac{8}{3+2}I_0\right)$$
$$= \frac{1024}{105}I_0$$

Since

$$I_0 = \int_0^4 x^0\sqrt{4-x}\,dx$$
$$= \int_0^4 \sqrt{4-x}\,dx$$
$$I_3 = \frac{1024}{105}\times\int_0^4 \sqrt{4-x}\,dx$$
$$= \frac{1024}{105}\left[-\frac{2}{3}(4-x)^{\frac{3}{2}}\right]_0^4$$
$$= -\frac{2048}{315}\left[(4-4)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right]$$
$$= -\frac{2048}{315}(-8)$$
$$= \frac{16\,384}{315}$$
$$= 52.0 \text{ (3 s.f.)}$$

13 a $I_n = \int \cos^n x \, dx$

$= \int \cos^{n-1} x \cos x \, dx$

Let $u = \cos^{n-1} x \Rightarrow \frac{du}{dx} = -(n-1)\cos^{n-2} x \sin x$

Let $\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$$\int \cos^{n-1} x \cos x \, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (\cos^{n-2} x - \cos^n x) dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \, dx - (n-1)\int \cos^n dx$$

Hence:

$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$nI_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$ as required

b $J_n = \int_0^{2\pi} \cos^n x \, dx$

$nJ_n = \left[\cos^{n-1} x \sin x\right]_0^{2\pi} + (n-1) J_{n-2}$

$= \cos^{n-1} 2\pi \sin 2\pi - \cos^{n-1} 0 \sin 0 + (n-1) J_{n-2}$

$= (n-1) J_{n-2}$

Therefore:

$J_n = \left(\frac{n-1}{n}\right) J_{n-2}$

c i $J_4 = \frac{3}{4} J_2$

$= \frac{3}{4}\left(\frac{1}{2} J_0\right)$

$= \frac{3}{8} J_0$

$= \frac{3}{8}\int_0^{2\pi} \cos^0 x \, dx$

$= \frac{3}{8}\int_0^{2\pi} dx$

$= \frac{3}{8}[x]_0^{2\pi}$

$= \frac{3}{8}(2\pi - 0)$

$= \frac{3}{4}\pi$

13 c ii $J_8 = \left(\frac{7}{8}\right)J_6$

$= \left(\frac{7}{8}\right)\left(\frac{5}{6}J_4\right)$

$= \frac{35}{48}J_4$

$= \frac{35}{48} \times \frac{3}{4}\pi$

$= \frac{35}{64}\pi$

d $J_n = \left(\frac{n-1}{n}\right)J_{n-2}$

When n is odd:

Let $n = 2k + 1$

$$J_{2k+1} = \left(\frac{(2k+1)-1}{2k+1}\right)\left(\frac{(2k+1)-3}{(2k+1)-2}\right)\cdot\ldots\cdot\int_0^{2\pi}\cos x\,\mathrm{d}x$$

$$= 0 \qquad \left(\text{since } \int_0^{2\pi}\cos x\,\mathrm{d}x = 0\right)$$

14 a $I_n = \int_0^1 x^n\left(1-x^2\right)^{\frac{1}{2}}\,dx = \int_0^1 x^{n-1}\left\{x\left(1-x^2\right)^{\frac{1}{2}}\right\}dx$

Let $u = x^{n-1} \Rightarrow \frac{du}{dx} = (n-1)x^{n-2}$

Let $\frac{dv}{dx} = x(1-x^2)^{\frac{1}{2}} \Rightarrow v = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}$

$I_n = \left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 + \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx$

$I_n = \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)^{\frac{3}{2}}\,dx$

$= \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)\left(1-x^2\right)^{\frac{1}{2}}\,dx$

$= \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)\left(1-x^2\right)^{\frac{1}{2}}\,dx$

$= \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)^{\frac{1}{2}}\,dx - \frac{n-1}{3}\int_0^1 x^{n-2}x^2\left(1-x^2\right)^{\frac{1}{2}}\,dx$

$= \frac{n-1}{3}\int_0^1 x^{n-2}\left(1-x^2\right)^{\frac{1}{2}}\,dx - \frac{n-1}{3}\int_0^1 x^n\left(1-x^2\right)^{\frac{1}{2}}\,dx$

Hence:

$I_n = \frac{n-1}{3}I_{n-2} - \frac{n-1}{3}I_n$

$I_n + \frac{n-1}{3}I_n = \frac{n-1}{3}I_{n-2}$

$I_n\left(1+\frac{n-1}{3}\right) = \frac{n-1}{3}I_{n-2}$

$I_n\left(\frac{n+2}{3}\right) = \frac{n-1}{3}I_{n-2}$

$\left(n+2\right)I_n = \left(n-1\right)I_{n-2}$ as required

14 b $I_n = \left(\frac{n-1}{n+2}\right) I_{n-2}$

$$I_7 = \frac{6}{9} I_5$$

$$= \frac{6}{9}\left(\frac{4}{7} I_3\right)$$

$$= \frac{8}{21} I_3$$

$$= \frac{8}{21}\left(\frac{2}{5} I_1\right)$$

$$= \frac{16}{105} I_1$$

$$= \frac{16}{105}\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$$

$$= \frac{16}{105}\left[-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= -\frac{16}{315}\left[\left(1-1^2\right)^{\frac{3}{2}} - \left(1-0^2\right)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{16}{315}$$

15 a $I_n = \int x^n \cosh x \, dx$

Let $u = x^n \Rightarrow \frac{du}{dx} = nx^{n-1}$

Let $\frac{dv}{dx} = \cosh x \Rightarrow v = \sinh x$

$\int x^n \cosh x \, dx = x^n \sinh x - n\int x^{n-1} \sinh x \, dx$

Consider $\int x^{n-1} \sinh x \, dx$:

Let $u = x^{n-1} \Rightarrow \frac{du}{dx} = (n-1)x^{n-2}$

Let $\frac{dv}{dx} = \sinh x \Rightarrow v = \cosh x$

$\int x^{n-1} \sinh x \, dx = x^{n-1} \cosh x - (n-1)\int x^{n-2} \cosh x \, dx$

Therefore:

$$\int x^n \cosh x \, dx = x^n \sinh x - n\left(x^{n-1} \cosh x - (n-1)\int x^{n-2} \cosh x \, dx\right)$$
$$= x^n \sinh x - nx^{n-1} \cosh x + n(n-1)\int x^{n-2} \cosh x \, dx$$

Hence:

$I_n = x^n \sinh x - nx^{n-1} \cosh x + n(n-1) I_{n-2}$ as required

b $I_4 = \int x^4 \cosh x \, dx$

$$= x^4 \sinh x - 4x^3 \cosh x + 12I_2$$
$$= x^4 \sinh x - 4x^3 \cosh x + 12\left(x^2 \sinh x - 2x \cosh x + 2I_0\right)$$
$$= x^4 \sinh x - 4x^3 \cosh x + 12x^2 \sinh x - 24x \cosh x + 24I_0$$
$$= \left(x^4 + 12x^2\right)\sinh x - 4\left(x^3 + 6x\right)\cosh x + 24I_0$$
$$= \left(x^4 + 12x^2\right)\sinh x - 4\left(x^3 + 6x\right)\cosh x + 24\int x^0 \cosh x \, dx$$
$$= \left(x^4 + 12x^2\right)\sinh x - 4\left(x^3 + 6x\right)\cosh x + 24\int \cosh x \, dx$$
$$= \left(x^4 + 12x^2\right)\sinh x - 4\left(x^3 + 6x\right)\cosh x + 24\sinh x + c$$
$$= \left(x^4 + 12x^2 + 24\right)\sinh x - 4\left(x^3 + 6x\right)\cosh x + c$$

15 c $I_n = x^n \sinh x - nx^{n-1}\cosh x + n(n-1)I_{n-2}$

Therefore:

$$I_3 = x^3 \sinh x - 3x^2 \cosh x + 6I_1$$
$$= x^3 \sinh x - 3x^2 \cosh x + 6\int x \cosh x \, dx$$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \cosh x \Rightarrow v = \sinh x$

$$\int x \cosh x \, dx = x \sinh x - \int \sinh x \, dx$$
$$= x \sinh x - \cosh x + c$$

$$I_3 = x^3 \sinh x - 3x^2 \cosh x + 6(x \sinh x - \cosh x) + c$$
$$= x^3 \sinh x - 3x^2 \cosh x + 6x \sinh x - 6\cosh x + c$$
$$= (x^3 + 6x)\sinh x - (3x^2 + 6)\cosh x + c$$

$$\int_0^1 x^3 \cosh x \, dx = \left[(x^3 + 6x)\sinh x - (3x^2 + 6)\cosh x\right]_0^1$$
$$= (1^3 + 6\times 1)\sinh 1 - (3\times 1^2 + 6)\cosh 1 + 6\cosh 0$$
$$= 7\sinh 1 - 9\cosh 1 + 6$$
$$= \frac{1}{2}\left(7(e - e^{-1}) - 9(e + e^{-1})\right) + 6$$
$$= \frac{1}{2}(-2e - 16e^{-1}) + 6$$
$$= -e - 8e^{-1} + 6$$
$$= -e - \frac{8}{e} + 6$$
$$= \frac{6e - e^2 - 8}{e}$$

16 a $I_n = \int \frac{\sin nx}{\sin x} dx$

$$I_{n-2} = \int \frac{\sin(n-2)x}{\sin x} dx$$

$$I_n - I_{n-2} = \int \frac{\sin nx}{\sin x} dx - \int \frac{\sin(n-2)x}{\sin x} dx$$
$$= \int \frac{\sin nx - \sin(n-2)x}{\sin x} dx$$
$$= \int \frac{2\cos\left(\frac{n+(n-2)}{2}\right)x \sin\left(\frac{n-(n-2)}{2}\right)x}{\sin x} dx$$
$$= 2\int \frac{\cos(n-1)x \sin x}{\sin x} dx$$
$$= 2\int \cos(n-1)x \, dx$$
$$= \frac{2}{n-1}\sin(n-1)x + c \text{ as required}$$

16 b i $I_4 - I_2 = \dfrac{2}{4-1}\sin(4-1)x$

$= \dfrac{2}{3}\sin 3x$

$I_4 = \dfrac{2}{3}\sin 3x + I_2$

$I_2 - I_0 = \dfrac{2}{2-1}\sin(2-1)x$

$= 2\sin x + c$

Therefore:

$I_4 = \dfrac{2}{3}\sin 3x + 2\sin x + c$

ii $I_5 - I_3 = \dfrac{2}{5-1}\sin(5-1)x = \dfrac{1}{2}\sin 4x$

$I_5 = \dfrac{1}{2}\sin 4x + I_3$

$I_3 - I_1 = \dfrac{2}{3-1}\sin(3-1)x$

$= \sin 2x$

$I_3 = \sin 2x + I_1$

Therefore:

$I_5 = \dfrac{1}{2}\sin 4x + \sin 2x + I_1$

$I_n = \int \dfrac{\sin nx}{\sin x}\,dx$

$I_1 = \int \dfrac{\sin x}{\sin x}\,dx$

Hence:

$I_5 = \dfrac{1}{2}\sin 4x + \sin 2x + \int dx$

$= \dfrac{1}{2}\sin 4x + \sin 2x + x + c$

So:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x}\,dx = \left[\frac{1}{2}\sin 4x + \sin 2x + x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{2}\sin\left(\frac{4\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) + \frac{\pi}{3}\right) - \left(\frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right) + \frac{\pi}{6}\right)$$

$$= \left(-\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) - \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

$$= \frac{3\pi - \sqrt{3}}{6}$$

17 a $I_n = \int \sinh^n x \, dx$

$= \int \sinh^{n-1} x \sinh x \, dx$

Let $u = \sinh^{n-1} x \Rightarrow \frac{du}{dx} = (n-1)\sinh^{n-2} x \cosh x$

Let $\frac{dv}{dx} = \sinh x \Rightarrow v = \cosh x$

$\int \sinh^{n-1} x \sinh x \, dx = \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x \cosh^2 x \, dx$

$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x (1 + \sinh^2 x) \, dx$

$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x \, dx - (n-1)\int \sinh^n x \, dx$

Hence:

$I_n = \sinh^{n-1} x \cosh x - (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1) I_n = \sinh^{n-1} x \cosh x - (n-1) I_{n-2}$

$nI_n = \sinh^{n-1} x \cosh x - (n-1) I_{n-2}$

17 b i $I_n = \frac{1}{n}\sinh^{n-1} x\cosh x - \left(\frac{n-1}{n}\right)I_{n-2}$

$$I_5 = \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{5}I_3$$
$$= \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{5}\left(\frac{1}{3}\sinh^2 x\cosh x - \frac{2}{3}I_1\right)$$
$$= \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\sinh^2 x\cosh x + \frac{8}{15}I_1$$
$$I_1 = \int \sinh^1 x\,\mathrm{d}x$$
$$= \int \sinh x\,\mathrm{d}x$$

Therefore:

$$I_5 = \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\sinh^2 x\cosh x + \frac{8}{15}\int \sinh x\,\mathrm{d}x$$
$$= \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\sinh^2 x\cosh x + \frac{8}{15}\cosh x + c$$

Hence:

$$\int_0^{\ln 3} \sinh^5 x\,\mathrm{d}x = \left[\frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\sinh^2 x\cosh x + \frac{8}{15}\cosh x\right]_0^{\ln 3}$$
$$= \left(\frac{1}{5}\sinh^4(\ln 3)\cosh(\ln 3) - \frac{4}{15}\sinh^2(\ln 3)\cosh(\ln 3) + \frac{8}{15}\cosh(\ln 3)\right)$$
$$- \left(\frac{1}{5}\sinh^4(0)\cosh(0) - \frac{4}{15}\sinh^2(0)\cosh(0) + \frac{8}{15}\cosh(0)\right)$$

$$\cosh(\ln 3) = \frac{1}{2}\left(\mathrm{e}^{\ln 3} + \mathrm{e}^{-\ln 3}\right) = \frac{5}{3}$$
$$\cosh(0) = \frac{1}{2}\left(\mathrm{e}^0 + \mathrm{e}^0\right) = 1$$
$$\sinh(\ln 3) = \frac{1}{2}\left(\mathrm{e}^{\ln 3} - \mathrm{e}^{-\ln 3}\right) = \frac{4}{3}$$
$$\sinh(0) = \frac{1}{2}\left(\mathrm{e}^0 - \mathrm{e}^0\right) = 0$$

Therefore:

$$\int_0^{\ln 3} \sinh^5 x\,\mathrm{d}x = \frac{1}{5}\times\left(\frac{4}{3}\right)^4\times\frac{5}{3} - \frac{4}{15}\times\left(\frac{4}{3}\right)^2\times\frac{5}{3} + \frac{8}{15}\times\frac{5}{3} - \frac{8}{15}$$
$$= \frac{1}{5}\times\frac{256}{81}\times\frac{5}{3} - \frac{4}{15}\times\frac{16}{9}\times\frac{5}{3} + \frac{8}{15}\times\frac{5}{3} - \frac{8}{15}$$
$$= \frac{256}{243} - \frac{64}{81} + \frac{40}{45} - \frac{8}{15}$$
$$= \frac{752}{1215}$$

17 b ii $I_n = \frac{1}{n}\sinh^{n-1} x \cosh x - \left(\frac{n-1}{n}\right) I_{n-2}$

$$I_4 = \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{4} I_2$$
$$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{4}\left(\frac{1}{2}\sinh x \cosh x - \frac{1}{2} I_0\right)$$
$$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8} I_0$$

Since:

$$I_0 = \int \sinh^0 x \, dx$$
$$= \int dx$$
$$I_4 = \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8}\int dx$$
$$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8}x$$

$$\cosh(\text{arsinh}1) = \sqrt{1+\sinh^2(\text{arsinh}1)} = \sqrt{1+1^2} = \sqrt{2}$$

Therefore:

$$\int_0^{\text{arsinh}1} \sinh^4 x \, dx = \left[\frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8}x\right]_0^{\text{arsinh}1}$$
$$= \frac{1}{4}\cosh(\text{arsinh}1) - \frac{3}{8}\cosh(\text{arsinh}1) + \frac{3}{8}(\text{arsinh}1) - 0$$
$$= \frac{1}{4}\cosh(\text{arsinh}1) - \frac{3}{8}\cosh(\text{arsinh}1) + \frac{3}{8}(\text{arsinh}1) - 0$$
$$= \frac{1}{4}\sqrt{2} - \frac{3}{8}\sqrt{2} + \frac{3}{8}\ln\left(1+\sqrt{1^2+1}\right)$$
$$= -\frac{\sqrt{2}}{8} + \frac{3}{8}\ln\left(1+\sqrt{2}\right)$$
$$= \frac{1}{8}\left(3\ln\left(1+\sqrt{2}\right) - \sqrt{2}\right) \text{ as required}$$

Challenge

a $I_n = \int x^a (\ln x)^n \, dx$

Let $u = (\ln x)^n \Rightarrow \frac{du}{dx} = \frac{n}{x}(\ln x)^{n-1}$

Let $\frac{dv}{dx} = x^a \Rightarrow v = \frac{x^{a+1}}{a+1}$

$$\int x^a (\ln x)^n \, dx = \frac{x^{a+1}}{a+1}(\ln x)^n - \int \frac{x^{a+1}}{a+1} \times \frac{n}{x}(\ln x)^{n-1} \, dx$$

$$= \frac{x^{a+1}}{a+1}(\ln x)^n - \frac{n}{a+1}\int \frac{x^{a+1}}{x} (\ln x)^{n-1} \, dx$$

$$= \frac{x^{a+1}}{a+1}(\ln x)^n - \frac{n}{a+1}\int x^a (\ln x)^{n-1} \, dx$$

Hence:

$$I_n = \frac{x^{a+1}}{a+1}(\ln x)^n - \frac{n}{a+1} I_{n-1}$$

b $I_n = \int x^a (\ln x)^n \, dx$

$I_3 = \int \sqrt{x} (\ln x)^3 \, dx$ where $a = \frac{1}{2}$

$I_n = \frac{2x^{\frac{3}{2}}}{3} (\ln x)^n - \frac{2n}{3} I_{n-1}$

Let $t = \ln x$

$I_n = \frac{2x^{\frac{3}{2}}}{3} t^n - \frac{2n}{3} I_{n-1}$

So:

$$I_3 = \frac{2x^{\frac{3}{2}}}{3} t^3 - \frac{6}{3} I_2$$

$$= \frac{2x^{\frac{3}{2}}}{3} t^3 - 2\left(\frac{2x^{\frac{3}{2}}}{3} t^2 - \frac{4}{3} I_1\right)$$

$$= \frac{2x^{\frac{3}{2}}}{3} t^3 - \frac{4x^{\frac{3}{2}}}{3} t^2 + \frac{8}{3} I_1$$

$$= \frac{2x^{\frac{3}{2}}}{3} t^3 - \frac{4x^{\frac{3}{2}}}{3} t^2 + \frac{8}{3}\left(\frac{2x^{\frac{3}{2}}}{3} t - \frac{2}{3} I_0\right)$$

$$= \frac{2x^{\frac{3}{2}}}{3} t^3 - \frac{4x^{\frac{3}{2}}}{3} t^2 + \frac{16x^{\frac{3}{2}}}{9} t - \frac{16}{9} I_0$$

$I_0 = \int \sqrt{x} (\ln x)^0 \, dx$

$= \int \sqrt{x} \, dx$

Therefore:

$$I_3 = \frac{2x^{\frac{3}{2}}}{3} t^3 - \frac{4x^{\frac{3}{2}}}{3} t^2 + \frac{16x^{\frac{3}{2}}}{9} t - \frac{16}{9} \int \sqrt{x} \, dx$$

$$= \frac{2x^{\frac{3}{2}}}{3} t^3 - \frac{4x^{\frac{3}{2}}}{3} t^2 + \frac{16x^{\frac{3}{2}}}{9} t - \frac{16}{9} \times \frac{2}{3} x^{\frac{3}{2}}$$

$$= \frac{2x^{\frac{3}{2}}}{3} t^3 - \frac{4x^{\frac{3}{2}}}{3} t^2 + \frac{16x^{\frac{3}{2}}}{9} t - \frac{32}{27} x^{\frac{3}{2}}$$

$$= \frac{18x^{\frac{3}{2}}}{27} t^3 - \frac{36x^{\frac{3}{2}}}{27} t^2 + \frac{48x^{\frac{3}{2}}}{27} t - \frac{32}{27} x^{\frac{3}{2}}$$

$$= \frac{2}{27} x^{\frac{3}{2}} \left(9t^3 - 18t^2 + 24t - 16\right)$$

Since $t = \ln x$:

$$I_3 = \frac{2}{27} x^{\frac{3}{2}} \left(9(\ln x)^3 - 18(\ln x)^2 + 24(\ln x) - 16\right) + c$$